The deletion Method.

The basic probabilistic method:

Trying to prove that a structure with certain desired property exists, one defines on appropriate probability space and then show the desired structure occurs with positive probability.

Today, we extend this and consider situations where "random" structure does not always have all the desired properties, but may have few blemishes. After removing these blemishes, we will obtain the desired structure.

- **Recall**: Ramsey $R(k,k)>n$ means that there is a 2-coloring of the edges of K_n by red and blue s.t. NO red K_k nor blue K_k.

Thm: If $\binom{n}{k}2^{1-\binom{k}{2}}<1$, then $R(k,k)>n$. (Consider a random 2-coloring)

Corollary: $R(k,k)>\frac{1}{\sqrt{2}e}k2^{\frac{k}{2}}$.

Thm: For $\forall$ integer n, $R(k,k)>n-\binom{n}{k}2^{1-\binom{k}{2}}$.

Pf: Consider a random 2-edge-coloring of K_n, where each edge is colored by red / blue with probability $\frac{1}{2}$, independent of other edges. For each $A\in\binom{[n]}{k}$, let X_A be the indicator random variable of the event that A is monochromatic (all edges red or blue). Let X be the number of monochromatic k-sets.

$$X=\sum_{A\in\binom{[n]}{k}}X_A.$$

Then, $$E(X)=\sum_{A\in\binom{[n]}{k}}E(X_A)=\sum_{A\in\binom{[n]}{k}}P(A\text{ is monochromatic})=\binom{n}{k}2^{1-\binom{k}{2}}.$$

Then there is a 2-edge-coloring s.t. its number of monochromatic k-sets $\leq\binom{n}{k}2^{1-\binom{k}{2}}$.

每个单色 K_k 都删一个点，可以破坏掉所有的 K_k. 那么 $n-\binom{n}{k}2^{1-\binom{k}{2}}$ 个点上就没有单色 K_k.

Fix such a 2-edge-coloring.

Remove one vertex from each monochromatic k-set to destroy this monochromatic k-set. After deleting at most $T = \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices, we can destroy all monochromatic k-sets. Now it remains s vertices, where $s \geq n - T$, which has NO monochromatic K_k.

Therefore, $R(k,k) > s \geq n - T = n - \binom{n}{k} 2^{1-\binom{k}{2}}$. #

Question: Find n to maximize $n - \binom{n}{k} 2^{1-\binom{k}{2}}$ for fixed k.

Corollary: $R(k,k) > \frac{1}{e}(1+o(1))\, k\, 2^{\frac{k}{2}}$. (Exercise)

// $> \frac{\sqrt{2}}{e}(1+o(1))\, k\, 2^{\frac{k}{2}}$.

HW: If $\exists\, p \in [0,1]$ s.t. $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$, then $R(k,l) > n$.

Exercise: For $\forall n$, $\forall p \in [0,1]$, $R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l}(1-p)^{\binom{l}{2}}$.

- Recall: Turán Thm: For $\forall G$, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+d(v)}$, where $\alpha(G) = \max_{\text{independent set } I} |I|$.

Corollary: If G has m edges and n vertices, then $\alpha(G) \geq \frac{n^2}{2m+n}$.

Write $m = \frac{nd}{2}$, then $\alpha(G) \geq \frac{n}{d+1}$. d: 平均度.

Next, we'll see a short (and nice) argument to show the halfway of Turán's Thm.

Thm: Let G be a graph with n vertices and $\frac{nd}{2}$ $(d>1)$ edges. Then $\alpha(G) \geq \frac{n}{2d}$.

Pf: Let $S \subseteq V(G)$ be a random set where $P_r(v \in S) = p$, where p will be chosen later. 边在S中: p^2. 破坏S中的边. $\geq np - \frac{nd}{2}p^2$.

Let $X = |S|$ and $Y = e(S)$.

For each edge $ij \in E(G)$, let Y_e be the indicator random variable for the event $ij \in S$. So, $Y = \sum_{e \in E(G)} Y_e$.

$\Rightarrow \mathbb{E}(Y) = \sum_{e\in E} \mathbb{E}(Y_e) = \sum_{e\in E} P_r(e\in S) = \sum_{e\in E} p^2 = p^2 \frac{nd}{2}$.

and $E(X) = np$.

$\Rightarrow E(X-Y) = np - \frac{nd}{2}p^2$.

So, there is a specific set S with $|S| - e(S) \geqslant np - \frac{nd}{2}p^2$.

Now we delete one vertex out of S for each edge in S. This leaves an independent set S^* with $|S^*| \geqslant np - \frac{nd}{2}p^2$ for $\forall\ p\in (0,1]$

By choosing $p = \frac{1}{d}$, $|S^*| \geqslant \frac{n}{2d}$.

$\Rightarrow \alpha(G) \geqslant |S^*| \geqslant \frac{n}{2d}$. #.

- Markov's inequality: Let $X \geqslant 0$ be a non-negative random variable and $t>0$. Then $P(X\geqslant t) \leq \frac{E(X)}{t}$.

  Pf: $\mathbb{E}X = \sum_{a\geqslant 0} a\, P_r(X=a) \geqslant \sum_{a\geqslant t} t\, P(X=a) = t\, Pr(X\geqslant t)$. #

Corollary: Let X_n be non-negtive integer-valued random variable in Probability space $(\Omega_n, \mathbb{P}_n)$. If $\mathbb{E}(X_n) \to 0$ (as $n\to+\infty$), then $Pr(X_n = 0) \to 1$ (as $n\to\infty$).

Pf: Choosing $t=1$ in Markov's Ineq., $Pr(X_n \geqslant 1) \leq \mathbb{E}(X_n) \to 0$. #

Thm: For a random $G(n,p)$ for some $0<p<1$, then

$$Pr\left(\alpha(G) \leq \left\lceil \frac{2\ln n}{p} \right\rceil\right) \to 1 \quad (\text{as } n\to+\infty)$$

OR almost surely $\alpha(G) \leq \left\lceil \frac{2\ln n}{p}\right\rceil$.

Pf: Let $k = \left\lceil\frac{2\ln n}{p}\right\rceil$, for every $S\in \binom{[n]}{k+1}$. Let A_S be the event that S is a stable set. Let X_S be the indicator random variable of A_S.

Let $X := \sum_{S\in\binom{[n]}{k+1}} X_S$ be the number of stable sets of size $k+1$.

We want to prove that $Pr(X_n = 0) \to 1$, and by corollary, it is enough to show $\mathbb{E}(X_n) \to 0$ (as $n\to+\infty$), as X_n is integer-valued.

$\mathbb{E}(X_n) = \sum_{S\in\binom{[n]}{k+1}} \mathbb{E}(X_S) = \sum_{S\in\binom{[n]}{k+1}} Pr(A_S) = \binom{n}{k+1}(1-p)^{\binom{k+1}{2}} \leq \frac{n^{k+1}}{(k+1)!} e^{-p\binom{k+1}{2}}$ $(1-p\leq e^{-p})$

$\leq \frac{1}{(k+1)!}\left(ne^{-\frac{pk}{2}}\right)^{k+1} \underset{k=\left\lceil\frac{2\ln n}{p}\right\rceil \geqslant \frac{2\ln n}{p}}{\leq} \frac{1}{(k+1)!} \to 0$ (as $n\to+\infty$).

Therefore, $\Pr(X_n=0) \to 1$ (as $n \to +\infty$).

Hence, almost surely NO sets of size $k+1$ is stable.

$\Rightarrow$ Almost surely $\alpha(G) \le k = \lceil \frac{2\ln n}{p} \rceil$. #

Def: For graph G, the chromatic number $\chi(G)$ is the minimum integer k such that $V(G)$ can be partitioned into k independent set.

Fact 1: If G has a complete graph K_k, then $\chi(G) \ge k$.

※ On the other hand, (quite surprisingly), there exists triangle-free graph with arbitrary high chromatic number. To show such graphs, we need

Fact 2: For any G with n vertices, $\chi(G) \cdot \alpha(G) \ge n$. $\Leftrightarrow \chi(G) \ge \frac{n}{\alpha(G)}$.

$\alpha(G) \ge$ $\chi(G)$个部分，每个部分的顶点个数.

Here, we prove an even stronger result.

Def: girth $g(G)$ is the length of a shortest cycle in G.

Thm: For any integers k and ℓ, there exists a graph G with $\chi(G) \ge k$ and $g(G) \ge \ell$. 取 $l = 4$ 可以证明 ※.

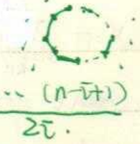
Thm 2: $\forall k$, $\exists G$ with $\chi(G) \ge k$ and $g(G) \ge k$.

Proof: Consider $G = G(n,p)$, where p will be decided later.

For $t := \lceil \frac{2\ln n}{p} \rceil$, we just show almost surely $\alpha(G) \le t$. ...①

有短圈就破坏掉。要让短圈很少。破坏之后 $\alpha(G)$ 可能变小，还是 $\le t$.

Let $X = \#$ cycles of length less than k.

K_n 

$$\Rightarrow E(X) = \sum_{i=3}^{k-1} \frac{n(n-1)\cdots(n-i+1)}{2i} \cdot p^i.$$

where $\frac{n(n-1)\cdots(n-i+1)}{2i}$ is the number of C_i in K_n, and each such cycle exists in $G(n,p)$ with probability p^i.

$\# C_i\text{'s} = \frac{n(n-1)\cdots(n-i+1)}{2i}$. 顺逆时针 圆排列

$$\Rightarrow E(X) \le \sum_{i=0}^{k-1} (np)^i = \frac{(np)^k - 1}{np - 1}$$

By Markov's Ineq, $\Pr(X \ge \frac{n}{2}) < \frac{E(X)}{\frac{n}{2}} \le \frac{2((np)^k - 1)}{n(np-1)}$

Choosing $p = n^{-\frac{k-1}{k}}$, $\Pr(X > \frac{n}{2}) < \frac{2(n-1)}{n(n^{\frac{1}{k}}-1)} \to 0$ (as $n \to +\infty$).

$\Leftrightarrow$ almost surely G has $\le \frac{n}{2}$ cycles of length $< k$. …②.

①+② $\Rightarrow$ for sufficiently large n, there exists a graph G on n vertices with $\alpha(G) \le t = \lceil \frac{2\ln n}{p} \rceil \le 3\ln n \cdot n^{\frac{k-1}{k}}$. and with $\le \frac{n}{2}$ cycles of length $< k$.

开始破坏短圈 (deleting method). 这里的 G 已 fixed.

By deleting one vertex from each cycles of length $< k$, we obtain a graph G' with at least $\frac{n}{2}$ vertices and with $g(G') \ge k$; moreover, $\alpha(G') \le \alpha(G) \le 3\ln n \cdot n^{\frac{k-1}{k}}$,

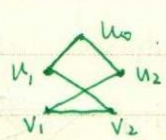
$\Rightarrow \chi(G') \ge \frac{|V(G')|}{\alpha(G')} \ge \frac{\frac{n}{2}}{3\ln n \, n^{\frac{k-1}{k}}} = \frac{n^{\frac{1}{k}}}{6\ln n}$.

Hence, by choosing n large enough, $\chi(G') \ge \frac{n^{\frac{1}{k}}}{6\ln n} \ge k$. #.

构造 Define the so-called Mycielski graph G_i as follows:

(1) $G_1 = \bullet$, $G_2 = \bullet\!-\!\bullet$;

(2). Given G_i on vertices $v_1, \cdots, v_n$. Let G_{i+1} be obtained from G_i by adding $u_1, \cdots, u_n, u_0$, s.t. $\begin{cases} \text{(R1) } u_j \text{ is adjacent to all neighbors of } v_j \text{ in } G_i. \\ \text{(R2) } u_0 \sim u_i \text{ for } 1 \le i \le n. \end{cases}$

$\Rightarrow G_3 =$ 

$= C_5$. $G_4 =$ 

= Grötzsh's graph.

Fact: Mycielski graph G_n is triangle-free and $\chi(G_n) = n$.

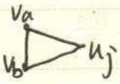
Def: $\chi(G) =$ min k s.t. $V(G) \to k$ independent sets.

Def: A proper coloring is a function $f: V(G) \to \mathbb{N}$, s.t. for any edge uv, $f(u) \ne f(v)$.

Def: $\chi(G) =$ min k s.t. $\exists$ a proper coloring $f: V(G) \to [k]$.

Proof: "Triangle-free":

By induction, for G_{i+1}, $\begin{cases} \cdot\ G_i \text{ is triangle free} \\ \cdot\ \{u_1, \cdots, u_n\} \text{ is an independent set} \end{cases}$

The only triangle in G_{i+1} would be 

where $j > 0$.
but this gives a triangle in G_i, a contradiction!

. (u_j与v_j的邻居相邻).

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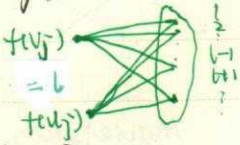
" $\chi(G_{i+1}) = i+1$ " :

(1). $\chi(G_{i+1}) \le i+1$: By induction, $\chi(G_i) = i$, that is, G_i have a proper coloring $f : \{v_1, \cdots v_n\} \to [i]$. Then, we can extend this f onto $\{u_1, \cdots u_n\}$ by assigning: $f(u_j) = f(v_j)$ for $1 \le j \le n$, $f(u_0) := i+1$.

(2). $\chi(G_{i+1}) > i$:

In other word, we should show that all functions $V(G_{i+1}) \to [i]$ can not be proper coloring. (反证) Suppose there is one, say f ...(*).
The the restriction of f on G_i is also a proper i-coloring.
Claim: If $\chi(G) = k$, and f is a proper k-coloring, then for $\forall 1 \le i \le k$, there is a vertex with color i adjacent to a vertex of every other color. i有所有其它颜色的邻点. (Prove by contradiction #).
(若∀i-点，与另一种颜色c(i)不相邻，可以把它们染成c(i). 那i就可以不用颜色i)

$\Rightarrow$ for $1 \le l \le i$, $\exists$ a vertex v_j with color l adjacent to a vertex of every other color.



By the definition, $f(u_j) = f(v_j) = l$ where $1 \le j \le n$.
This shows that we need to use all colors $1, 2, \cdots i$ to color $u_1, u_2, \cdots u_n$. But u_0 is adjacent to $u_1, \cdots u_n$. There is NO space for u_0.

$\Rightarrow$ This shows that (*) fails and $\chi(G_{i+1}) > i$.

(1) + (2) $\Rightarrow$ $\chi(G_{i+1}) = i+1$.

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